VISUALIS EXEMPLE SCENES - 3-BODY PROBLEM

FRANÇOIS SAHY, VISUALIS DYNAMICS TUTORIALS

Abstract. This document is part of Visualis help, and explores the well known 3-body problem, presenting the examples available in Visualis Dynamics, along with mathematical derivations. Visualis offers an simple platform for simulating many college level problems, making it easier to verify results and deepen understanding through hands-on learning.

1. Well known chaotic problem

The 3-body problem is a classic problem in celestial mechanics, involving prediction of the motion of three (or more) planets moving under the influence of their mutual gravitation. This scenario is far more complex than the two-body problem, which has a straightforward, predictable solution, allowing for precise predictions of motions.

1.1. Difficult to Solve Analytically.

We can highlight 3 main ideas :

Non-linearity of Gravitational Interactions : The equations governing the gravitational interactions between three masses are highly non-linear. This makes it impossible to find a general, closed-form solution that describes the motion of the planets over time.

Sensitivity to Initial Conditions : The three-body system is highly sensitive to the exact initial conditions (positions and velocities, even masses). Any tiny change in the initial setup can lead to vastly different outcomes. This sensitivity (known as the "butterfly effect") means that even if we had a general set of solutions, they would be practically useless without knowing exact initial conditions.

Infinite Number of Possible Configurations : With three bodies, the number of possible configurations is infinite, depending on masses, start position and velocities. This is very different from a 2 bodies/planets configuration. This infinite possibility space further complicates finding any one-size-fits-all analytical solution.

1.2. **Indicators of Chaos.** In other terms :

Chaotic Behavior : The three-body problem is a perfect example of a chaotic system, where the trajectories of the planets can become unpredictably erratic over time, especially in configurations where the masses can get closer to each other, due to $\frac{1}{r^2}$ ratio. So out of some very precise initial conditions, a gravitational system with more than 2 planets is characterized by a lack of long-term predictability.

Initial Conditions : As said a hallmark of chaotic systems is their extreme sensitive dependence on initial conditions. Arbitrarily small changes in the starting state of the system will lead to completely different outcomes, preventing an accurate closed-form solution.

Lack of Repeatable Patterns : The trajectories of the bodies do not settle into stable, repeatable patterns (except in very special or contrived cases). In most cases their paths will appear erratic, until reaching a simplified 2-body-like configuration.

1.3. **Stability in the Universe.**

Despite the inherent chaos and unpredictability in this problem, stable configurations obviously do exist in the universe ! These configurations rely on certain ratios of masses and distances that reduce the system's sensitivity to initial conditions. For example:

Hierarchical Systems : Systems where two planets are close together and the third is much farther away can approximate a stable two-body problem for the close pair, with the distant mass having a minimal perturbative effect. This configuration can remain stable over millions of years.

Resonant Orbits : In some cases, planets can fall into resonant orbits, where their orbital periods form simple ratios. This resonance can help stabilize the system by preventing close encounters that could disrupt the orbits.

Special Solutions : Certain specific solutions to the three-body problem, known as Lagrange points and the non-intuitive figure-eight orbit (discovered numerically) [4], demonstrate stability under specific conditions.

In the Universe, stable configurations emerged through a process of "natural selection", where unstable arrangements eventually disappeared over time.

2. Examples in Visualis

As said before some special cases can be derived analytically, let's explore two examples.

2.1. **Semi-stable but highly chaotic 3 masses.** 3-planets.vis file

This one is very theoretical, highly unstable, so you'll never find it in real world ! We consider 3 perfectly equal masses that rotate in a circular orbit while maintaining a perfectly equilateral triangle configuration. It's known as "Lagrange's equilateral triangle homothetic solution".

For our case we can consider that the three equal masses M are known, as well as their distance, and we can try to find the stable orbital velocity v required to maintain this configuration. (And we want to derive the equations of motion from Newton's law).

2.1.1. *Coordinate System and Initial Setup.*

Center of Mass :

Since all masses are equal, the center of mass C of the system is at the centroid of the equilateral triangle.

Positions of the Masses :

We place the masses at positions that rotate uniformly around C with angular velocity ω. At any time t :

(1) Mass M_1 :

$$\vec{r}_1(t) = r\cos(\omega t)\hat{i} + r\sin(\omega t)\hat{j}$$

(2) Mass M_2 :

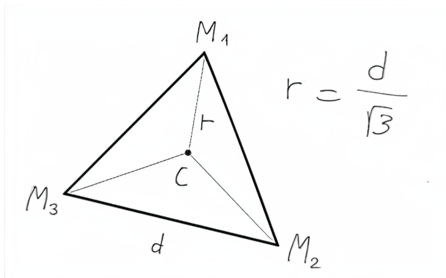
$$\vec{r}_2(t) = r\cos\left(\omega t + \frac{2\pi}{3}\right)\hat{i} + r\sin\left(\omega t + \frac{2\pi}{3}\right)\hat{j}$$

(3) Mass M_3 :

$$\vec{r}_3(t) = r\cos\left(\omega t + \frac{4\pi}{3}\right)\hat{i} + r\sin\left(\omega t + \frac{4\pi}{3}\right)\hat{j}$$

where r is the distance from each mass to the center C.

Relationship Between r and Side Length d :



In an equilateral triangle :

$$r = \frac{d}{\sqrt{3}}$$

Gravitational Force Between Two Masses

The gravitational force between any two masses M separated by d is :

$$F = G\frac{M^2}{d^2}$$

Net Gravitational Force on Mass M_1

We need to find the vector sum of the gravitational forces acting on M_1 due to M_2 and M_3.

Positions at $t = 0$: $\vec{r}_1 = (r, 0)$, $\vec{r}_2 = r\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, $\vec{r}_3 = r\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

Vectors from M_1 to M_2 and M_3 :

$$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1 = r\left(-\frac{3}{2}, \frac{\sqrt{3}}{2}\right)$$

$$\vec{r}_{13} = \vec{r}_3 - \vec{r}_1 = r\left(-\frac{3}{2}, -\frac{\sqrt{3}}{2}\right)$$

Unit Vectors :

$$\hat{r}_{12} = \frac{\vec{r}_{12}}{|\vec{r}_{12}|} = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

$$\hat{r}_{13} = \frac{\vec{r}_{13}}{|\vec{r}_{13}|} = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

Gravitational Forces :

$$\vec{F}_{12} = G\frac{M^2}{|\vec{r}_{12}|^2}\hat{r}_{12} = G\frac{M^2}{3r^2}\hat{r}_{12}$$

$$\vec{F}_{13} = G\frac{M^2}{|\vec{r}_{13}|^2}\hat{r}_{13} = G\frac{M^2}{3r^2}\hat{r}_{13}$$

Net Gravitational Force on M_1 :

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} = G\frac{M^2}{3r^2}\left(\hat{r}_{12} + \hat{r}_{13}\right)$$

Calculating $\hat{r}_{12} + \hat{r}_{13}$:

$$\hat{r}_{12} + \hat{r}_{13} = \left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}, \frac{1}{2} - \frac{1}{2}\right) = \left(-\sqrt{3}, 0\right)$$

Net Force :

$$\vec{F}_1 = G\frac{M^2}{3r^2}\left(-\sqrt{3}, 0\right)$$

Since the force is attractive :

$$\vec{F}_1 = -G\frac{M^2}{3r^2}\left(-\sqrt{3}, 0\right) = G\frac{M^2}{3r^2}\sqrt{3}\hat{\imath}$$

Simplify :

$$F_{1x} = G\frac{M^2\sqrt{3}}{3r^2}$$

Centripetal Force and Equations of Motion :

Mass M_1 moves in a circle of radius r with angular velocity ω :

$$a_{\text{centripetal}} = r\omega^2$$

Newton's Second Law :

Set the net gravitational force equal to the centripetal force required for circular motion :

$$Ma_{\text{centripetal}} = F_{1x}$$

$$Mr\omega^2 = G\frac{M^2\sqrt{3}}{3r^2}$$

Simplify :

$$r\omega^2 = G\frac{M\sqrt{3}}{3r^2}$$

Solving for Angular Velocity ω :

$$\omega^2 = \frac{GM\sqrt{3}}{3r^3}$$

Since $\frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$:

$$\omega = \sqrt{\frac{GM}{\sqrt{3}r^3}}$$

Solving for Linear Velocity v :

The linear velocity v is : $v = r\omega$

Substitute ω :

$$v = r\sqrt{\frac{GM}{\sqrt{3}r^3}} = \sqrt{\frac{GM}{\sqrt{3}r}}$$

Final Result

The stable linear velocity v required for each mass to maintain the rotating equilateral triangle configuration is :

$$v = \sqrt{\frac{1}{\sqrt{3}}\frac{GM}{r}}$$

But $d = \sqrt{3}\,r$. Hence $r = d/\sqrt{3}$. Substitute $r = d/\sqrt{3}$:

$$v = \sqrt{\frac{1}{\sqrt{3}}\frac{GM}{(d/\sqrt{3})}} = \sqrt{\frac{\sqrt{3}\,G\,M}{3\,(d/\sqrt{3})}} = \sqrt{\frac{\sqrt{3}\,G\,M\,\sqrt{3}}{3\,d}} = \sqrt{\frac{3\,G\,M}{3\,d}} = \sqrt{\frac{G\,M}{d}}.$$

Therefore, for three equal masses forming an equilateral triangle of side d, each mass orbits with the velocity :

$$\boxed{v = \sqrt{\frac{G\,M}{d}}}$$

And the angular velocity is :

$$\omega = \sqrt{\frac{3\,G\,M}{d^3}} \quad \text{with} \quad v = r\,\omega$$

Position as a function of time :

$$\vec{r}_i(t) = r\cos(\omega t + \theta_i)\,\hat{i} + r\sin(\omega t + \theta_i)\,\hat{j}$$

where $\theta_i = 0,\ \frac{2\pi}{3},\ \frac{4\pi}{3}$ for M_1, M_2, and M_3, respectively.

2.2. **Stable Hierarchical Systems example.** hierarchical 3-planets.vis

Let's consider the following more realistic, almost stable case :

- Three masses M_1, M_2, and M_3 are all on (or start on) a common line (the x-axis, say).
- The inner binary (masses M_1 and M_2) is in a stable, circular orbit about their common center of mass.
- The outer mass M_3 is on a much larger quasi-circular orbit around the center of mass of M_1 and M_2.
- The mass M_3 is relatively small compared to M_1 and M_2 so its influence on the inner binary orbit is negligible to first order.

With these assumptions, we can approximate the three-body configuration as a hierarchical triple system :

- The inner binary acts like a two-body problem.
- The outer mass M_3 orbits the combined mass of M_1 and M_2 at a large distance.

Step 1 : Inner Binary Circular Orbit

Setup for the Inner Binary :

- Let the distance between M_1 and M_2 be a.
- Without loss of generality, place the center of mass (CM) of the system at the origin.
- Let $M = M_1 + M_2$.

The center of mass condition is :

$$M_1 x_1 + M_2 x_2 = 0.$$

If we choose a coordinate system so that M_1 and M_2 are initially aligned along the x-axis and orbit in the xy-plane, we have :

$$x_1 = -\frac{M_2}{M} a, \quad x_2 = \frac{M_1}{M} a$$

Since they are in a circular orbit around their common center of mass, each mass experiences centripetal force provided by the gravitational attraction of the other. For M_1 :

$$\text{Centripetal force on } M_1 = M_1 a_1 \omega_{12}^2$$

where $a_1 = \frac{M_2}{M} a$ is the orbital radius of M_1 about the CM. The gravitational force on M_1 from M_2 is :

$$F_{12} = \frac{G M_1 M_2}{a^2}$$

Equating centripetal and gravitational forces :

$$M_1 a_1 \omega_{12}^2 = \frac{G M_1 M_2}{a^2}$$

Canceling M_1 and substituting $a_1 = \frac{M_2}{M} a$:

$$\frac{M_2}{M} a \omega_{12}^2 = \frac{G M_2}{a^2} \implies \omega_{12}^2 = \frac{G(M_1 + M_2)}{a^3}$$

Thus, the angular velocity of the inner binary is :

$$\omega_{12} = \sqrt{\frac{G(M_1 + M_2)}{a^3}}$$

Parametric Equations for the Inner Binary :

We can write the positions as functions of time :

$$x_1(t) = -\frac{M_2}{M} a \cos(\omega_{12} t), \quad y_1(t) = -\frac{M_2}{M} a \sin(\omega_{12} t),$$
$$x_2(t) = \frac{M_1}{M} a \cos(\omega_{12} t), \quad y_2(t) = \frac{M_1}{M} a \sin(\omega_{12} t).$$

Step 2 : Outer Orbit of M_3

Now, consider the small mass M_3 orbiting at a large distance R from the CM of $M_1 + M_2$. Because M_3 is far away, the inner binary appears as a single mass $M = M_1 + M_2$ located at the origin (to a good approximation, ignoring small perturbations).

For a circular orbit of M_3 around mass M :

$$\frac{G(M_1 + M_2) M_3}{R^2} = M_3 R \omega_3^2$$

Canceling M_3 :

$$\omega_3^2 = \frac{G(M_1 + M_2)}{R^3}$$

Thus :

$$\omega_3 = \sqrt{\frac{G(M_1 + M_2)}{R^3}}$$

Parametric Equations for M_3 : assuming M_3 also starts on the positive x-axis and orbits in the same plane :

$$x_3(t) = R\cos(\omega_3 t), \quad y_3(t) = R\sin(\omega_3 t).$$

Step 3 : Combine the Motions

We now have a set of stable, closed-form solutions for the positions of all three masses assuming circular orbits :

Inner Binary :

$$x_1(t) = -\frac{M_2}{M} a\cos(\omega_{12} t), \quad y_1(t) = -\frac{M_2}{M} a\sin(\omega_{12} t),$$

$$x_2(t) = \frac{M_1}{M} a\cos(\omega_{12} t), \quad y_2(t) = \frac{M_1}{M} a\sin(\omega_{12} t).$$

Outer Mass :

$$x_3(t) = R\cos(\omega_3 t), \quad y_3(t) = R\sin(\omega_3 t).$$

Given the masses and the distances a (inner separation) and R (outer orbital radius), we have determined the angular velocities ω_{12} and ω_3 directly from Newton's law of gravitation for circular orbits. This set of equations provides a stable, analytic representation of the system's motion under the stated assumptions.

2.2.1. *Notes on Stability and Perturbations.* The solution is based on strict circular orbits and neglects the back-reaction of M_3 on the inner binary. In practice, there will be perturbations. If M_3 is truly small compared to M_1 and M_2, these perturbations remain small, and the motion is quasi-stable for very long times. More complicated analytical forms exist for hierarchical triples, but the above provides a simple first-order approximation.

2.2.2. *Summary.* By assuming that the system consists of an inner binary in a perfectly circular orbit and an outer mass M_3 orbiting at a large distance, we reduce the three-planets problem to a pair of nested two-planets problems. Under these conditions, we can use the standard formulae for circular orbital motion to write down exact expressions for the angular velocities and positions as functions of time. This yields a analytical equation of motion for all three planets, given their masses and orbital radii.

2.2.3. *Compute numerical values for Visualis Example.* Let's go through an example of how to assign initial velocities to the given system. In Visualis scene we have :

- The two big masses M_1 and M_2 (1×10^{23} kg each), located at $(0, -5 \times 10^6)$ and $(0, +5 \times 10^6)$ respectively, orbit (approximately) about their mutual center in a near-circular "inner binary".
- The smaller mass M_3 (1×10^{21} kg) at $(0, +6 \times 10^7)$ orbits (approximately) on a larger circle around the combined mass of $M_1 + M_2$.

We will ignore any small residual drift of the true center of mass (CM) and also neglect the slight pull of M_3 on M_1 and M_2. Under these simplifying assumptions, we get a quasi-stable triple system.

The Basic Configuration

- Masses : $M_1 = M_2 = 1.0 \times 10^{23}\,\text{kg}, \quad M_3 = 1.0 \times 10^{21}\,\text{kg}$
- Initial Positions (in meters, all on the y-axis) :

  $M_1 : (0,\ -5 \times 10^6), \quad M_2 : (0,\ +5 \times 10^6), \quad M_3 : (0,\ +6 \times 10^7)$
- Velocities (in m/s, all on the x-axis) : M_1 going left (negative x) so it circles clockwise if viewed from above, M_2 going right (positive x), M_3 going right (positive x).

We will treat $(0, 0)$ as an approximate center for the inner binary and let M_3 orbit further out. Strictly, the true center of mass is a bit above $(0, 0)$, but because $M_3 \ll M_1, M_2$, the shift is small enough that the following circular speeds still give a decent first approximation.

Inner Binary $M_1 - M_2$

For two equal masses $M_1 = M_2$ separated by $a = 1 \times 10^7$ m, the usual circular-orbit formula (about their midpoint) gives the orbital angular velocity

$$\omega_{12} = \sqrt{\frac{G\,(M_1 + M_2)}{a^3}} = \sqrt{\frac{G\,(2M_1)}{(1 \times 10^7)^3}} \quad \text{(since } M_1 = M_2\text{)}$$

With :

$$G \approx 6.6743 \times 10^{-11}, \quad M_1 = 1.0 \times 10^{23}, \quad a = 1.0 \times 10^7$$

we get :

$$\omega_{12}^2 = 6.6743 \times 10^{-11} \times \frac{2 \times 10^{23}}{(1 \times 10^7)^3} = 6.6743 \times 10^{-11} \times \frac{2 \times 10^{23}}{10^{21}} = 1.33486 \times 10^{13-11-21} = 1.33486 \times 10^{-8}$$

$$\omega_{12} \approx \sqrt{1.33486 \times 10^{-8}} \approx 1.155 \times 10^{-4}\,\text{rad/s}$$

Each mass is half the distance $a/2 = 5 \times 10^6$ from the midpoint. Thus the circular speed for each big mass is

$$v_{12} = \omega_{12} \times \frac{a}{2} = (1.155 \times 10^{-4}) \times (5 \times 10^6) \approx 578\,\text{m/s}$$

In the presence of M_3, the "true" center of mass is slightly above the midpoint of M_1 and M_2. In fact the actual center of mass is at :

$$y_{\mathrm{CM}} = \frac{-5\times10^6 + 5\times10^6 + 6\times10^7}{M_1+M_2+M_3} \approx \frac{0+6\times10^{28}}{2\times10^{23}+1\times10^{21}} \approx 3\times10^5\,\mathrm{m}.$$

That is only 300 km above the midpoint, compared to a 5,000,000 m separation on each side for the big masses.

Hence for M_1, its actual orbital radius about the "true" CM is about $5{,}000{,}000 + 300{,}000 = 5.3\times10^6$ m, and similarly 4.7×10^6 m for M_2.

Recomputing the centripetal speeds with extra 0.3 Mm distance yields :

- M_1: $v_1 \approx 590-600\,\mathrm{m/s}$ (to the left)
- M_2: $v_2 \approx 560\,\mathrm{ish\,m/s}$ (to the right)

So in the simulator we should get roughly :

$$\boxed{v_1 \approx -595\,\mathrm{m/s}} \quad \text{and} \quad \boxed{v_2 \approx +560\,\mathrm{m/s}}$$

Outer Orbit M_3

Now treat $M_1 + M_2 \approx 2\times10^{23}$ kg as a single central mass for M_3. Its distance from that (approximate) center is $\approx 6\times10^7 - 3\times10^5 = 5.97\times10^7$ m.

By Kepler's third law for a small body orbiting mass $(M_1 + M_2)$:

$$\omega_3^2 = \frac{G\,(M_1+M_2)}{r_3^3},$$

$$v_3 = \omega_3\,r_3 = \sqrt{\frac{G\,(M_1+M_2)}{r_3}}.$$

Numerically, with $M_1 + M_2 = 2\times10^{23}$ kg and $r_3 = 5.97\times10^7$ m :

The gravitational parameter : $G(M_1+M_2) = 6.6743\times10^{-11}\times2\times10^{23} \approx 1.33486\times10^{13}$.

Hence :

$$v_3 = \sqrt{\frac{1.33486\times10^{13}}{5.97\times10^7}} = \sqrt{2.235\times10^5} \approx 473\,\mathrm{m/s}.$$

(direction : x positive, i.e. to the right, same sense of rotation as M_2)

So a quite good approximate outer velocity for a stable orbit is :

$$\boxed{v_3 \approx +473\,\mathrm{m/s}}$$

Final Approximate Velocities

Putting it all together (and rounding slightly):

- M_1 at $(0,\, -5 \times 10^6) :\, \mathbf{v}_1 \approx (-595,\, 0)\,\mathrm{m/s}$
- M_2 at $(0,\, +5 \times 10^6) :\, \mathbf{v}_2 \approx (+560,\, 0)\,\mathrm{m/s}$
- M_3 at $(0,\, +6 \times 10^7) :\, \mathbf{v}_3 \approx (+473,\, 0)\,\mathrm{m/s}$

With these values:

- M_1 and M_2 form a near-circular inner binary.
- M_3 orbits farther out, as if around a central mass of $M_1 + M_2$.
- Because M_3 is $100\times$ lighter, its perturbation on the inner binary is small, and the system can remain in a "quasi-stable" configuration for quite a long time (though not perfectly forever, since true three-body motion always introduces some perturbations).

To go further we should fine-tune or numerically integrate the full three-body equations to get the best initial conditions to keep the center of mass as stationary as possible and minimize orbital drift. But a first evaluation/overview, the above velocities are a good first-order solution.

References

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Email address: support@visualis-physics.com

URL: http://www.visualis-physics.com