VISUALIS EXEMPLE - GRAVITY INSIDE SPHERE

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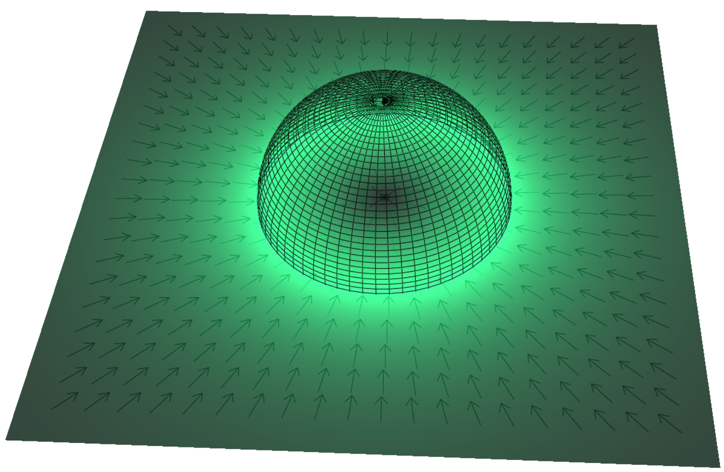
1. Gravity inside a massive sphere

The gravity inside a massive object decreases as we approach its center. This sample scene in Visualis shows the intensity of the gravitational field around and inside a massive sphere ; we see the values reaching zero when approaching the center.
(Note : the density there is nonetheless very high !)



At the sphere's center, the gravitational field is intuitively null because we are attracted equally by all points of the spherical mass around us. For other locations, the intensity can be evaluated by integrating the contribution of each mass element :

$$\vec{g}_P = -G \int_{V_S} \frac{\vec{r}_P - \vec{r}'}{|\vec{r}_P - \vec{r}'|} \cdot \frac{dm}{|\vec{r}_P - \vec{r}'|^2} = -G \int_{V_S} \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \rho(\vec{r}') dV'$$

where :

- G is the gravitational constant,
- $\vec{r}$ is the position vector of point P where we evaluate the field,
- $\vec{r}'$ is the position vector of the mass element dm,
- ρ is the mass density,
- V_S is the volume of the sphere.

1.1. **Direct Integration.**

We'll try here to derive the gravitational field inside a uniform sphere of radius R and mass M by direct integration from Newton's law !

1.1.1. *Problem Setup.*

For a point P at distance r from the sphere's center (where $r < R$), we place P on the z-axis at position $(0, 0, r)$ by symmetry.

The gravitational field from a mass element dm at position $\vec{r'}$ is :

$$d\vec{g} = -G\frac{dm}{|\vec{r}_P - \vec{r'}|^2} \cdot \frac{\vec{r}_P - \vec{r'}}{|\vec{r}_P - \vec{r'}|} \tag{1}$$

We assume uniform density $\Rightarrow \rho = \frac{3M}{4\pi R^3}$:

$$d\vec{g} = -G\rho\frac{\vec{r}_P - \vec{r'}}{|\vec{r}_P - \vec{r'}|^3}dV' \tag{2}$$

1.1.2. *The Integral in Spherical Coordinates.*

To use the symmetry we need to switch to spherical coordinates and perform a volume integral.

With $\vec{r}_P = (0, 0, r)$ and spherical coordinates for $\vec{r'}$:

$$|\vec{r}_P - \vec{r'}| = \sqrt{r^2 + r'^2 - 2rr'\cos\theta'} \tag{3}$$

By symmetry, only the z-component survives :

$$g_z = -2\pi G\rho \int_0^R r'^2 \int_0^\pi \frac{(r - r'\cos\theta')\sin\theta'}{(r^2 + r'^2 - 2rr'\cos\theta')^{3/2}}\, d\theta' dr' \tag{4}$$

1.1.3. *Evaluating the Angular Integral.*

This is a non-trivial integral because of singularities, but it can be done using substitutions.

Substituting $u = \cos\theta'$:

$$I(r, r') = \int_{-1}^{1} \frac{r - r'u}{(r^2 + r'^2 - 2rr'u)^{3/2}}du \tag{5}$$

Let $w(u) = r^2 + r'^2 - 2rr'u$, so $\frac{dw}{du} = -2rr'$,

and consider the auxiliary function :

$$F(u) = \frac{u - r'/r}{\sqrt{w(u)}} \tag{6}$$

Differentiating :

$$\frac{dF}{du} = \frac{w(u) + rr'\left(u - \frac{r'}{r}\right)}{w(u)^{3/2}} = \frac{r(r - r'u)}{w^{3/2}} \tag{7}$$

Therefore the integral becomes :

$$I(r, r') = \frac{1}{r}[F(1) - F(-1)] \tag{8}$$

For $r' < r$:

$$F(1) = \frac{1 - r'/r}{r - r'} \tag{9}$$
$$F(-1) = \frac{-1 - r'/r}{r + r'}$$

After some algebra : $F(1) - F(-1) = \frac{2}{r}$, so $I(r, r') = \frac{2}{r^2}$.

For $r' > r$: The calculation gives $F(1) - F(-1) = 0$, so $I(r, r') = 0$.

And therefore :

$$I(r, r') = \begin{cases} \frac{2}{r^2} & \text{if } r' < r \\ 0 & \text{if } r' > r \end{cases} \tag{10}$$

1.1.4. *Completing the Radial Integration.*

Since the outer region contributes zero :

$$g_z = -2\pi G\rho \int_0^r r'^2 \cdot \frac{2}{r^2} dr' \tag{11}$$

$$g_z = -\frac{4\pi G\rho}{r^2} \int_0^r r'^2 dr' = -\frac{4\pi G\rho}{r^2} \cdot \frac{r^3}{3} = -\frac{4\pi G\rho}{3} r$$

Substituting $\rho = \frac{3M}{4\pi R^3}$:

$$\boxed{g_z = -\frac{GMr}{R^3}} \tag{12}$$

(The negative sign indicates the field points inward)

So a simple linear decrease of the field as reaching the center !

1.2. **Key Insights.**

1. The direct integration naturally proves the Shell Theorem : mass outside radius r contributes zero to gravitation field at point P !

2. The tricky step is to correctly evaluating the angular integral using the auxiliary function $F(u) = \frac{u - r'/r}{\sqrt{w}}$, which avoids many complexity.

1.2.1. *Intuitive reasoning.*

We can also find this result by simple geometrical reasoning. As said earlier when considering symmetries we know that the gravitation field at the center will be null. Since we're convinced of this, as we also know that gravity at the sphere surface will be $\frac{GM}{R^2}$, we can suppose the simple linear ratio along distance r :

$$I_p = -\frac{GM}{R^2} \cdot \frac{r}{R} = -\frac{GMr}{R^3}$$

Which is the same result as above !

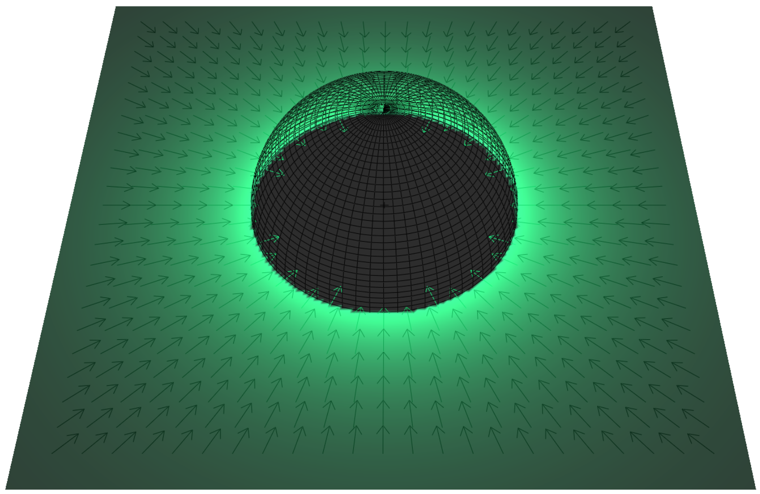
(Recall : this is an approximation, for a very massive body like a planet the matter is not uniformly distributed ; the density will be higher at center, so as one moves away the gravity will increase faster at first, and then slower than with the linear model of a uniform mass.)

But even with non-linear density, for points inside the sphere, we see that everything works as if only the mass at a radius less than r contributes to the gravitational field at that point !

Which lead us once again to the Shell Theorem !

2. Case of a spherical shell

For a hollow spherical shell where all mass M is distributed on the surface at radius R, we can see in Visualis that we have :



So no gravity at all inside the sphere (!) The Shell Theorem gives us :

$$g(r) = \begin{cases} 0 & \text{for } r < R \text{ (inside the shell)} \\ \frac{GM}{r^2} & \text{for } r \geq R \text{ (outside the shell)} \end{cases}$$

2.1. **Demonstration.**

2.1.1. *Integral on sphere surface.*

Using spherical coordinates centered at the shell's center, with P along the z-axis at distance r from center :

The force from element $dA = R^2 \sin\theta \, d\theta \, d\phi$ is :

$$d\vec{F} = \frac{G\sigma R^2 \sin\theta \, d\theta \, d\phi}{s^2} \hat{s} \tag{1}$$

where s is the distance from the element to P :

$$s = \sqrt{r^2 + R^2 - 2rR\cos\theta}$$

By symmetry, only the z-component survives.

We have the cosine of the angle between $d\vec{F}$ and the axis : $\cos\alpha = \dfrac{R\cos\theta - r}{s}$

So the z-component of force is :

$$dF_z = \frac{G\sigma R^2 \sin\theta \, d\theta \, d\phi}{s^2} \cdot \frac{R\cos\theta - r}{s} \tag{2}$$

Integrating over the sphere :

$$F_z = 2\pi G\sigma R^2 \int_0^\pi \frac{\sin\theta(R\cos\theta - r)}{s^3} \, d\theta \tag{3}$$

Using substitution $x = \cos\theta$, $dx = -\sin\theta \, d\theta$:

$$F_z = 2\pi G\sigma R^2 \int_{-1}^{1} \frac{Rx - r}{(r^2 + R^2 - 2rRx)^{3/2}} \, dx \tag{4}$$

This integral evaluates to zero !

2.1.2. *Resolution in detail.*

Let $u = r^2 + R^2 - 2rRx \;\; (> 0)$. Then $dx = -\,du/(2rR)$ and $Rx - r = \dfrac{R^2 - r^2 - u}{2r}$.

The limits $x = \pm 1$ map to $u = (r \mp R)^2$. With those substitutions,

$$F_z = \frac{\pi G\sigma R}{2r^2} \int_{(r-R)^2}^{(r+R)^2} \frac{R^2 - r^2 - u}{u^{3/2}} \, du \tag{5}$$

Split and integrate term-by-term :

$$\int \frac{R^2 - r^2}{u^{3/2}} \, du \;=\; -\frac{2(R^2 - r^2)}{\sqrt{u}}, \qquad \int \frac{u}{u^{3/2}} \, du \;=\; 2\sqrt{u}. \tag{6}$$

Putting it together,

$$F_z = \frac{\pi G\sigma R}{2r^2} \Big[-2(R^2 - r^2)u^{-1/2} - 2u^{1/2} \Big]_{(r-R)^2}^{(r+R)^2} \tag{7}$$

Define $f(u) := (R^2 - r^2)u^{-1/2} + u^{1/2}$

Then we have :

$$F_z = -\frac{\pi G\sigma R}{r^2} \Big[f\big((r+R)^2\big) - f\big((r-R)^2\big) \Big]. \tag{8}$$

Evaluate f at the two limits :

$$\begin{aligned} f\big((r+R)^2\big) &= (R^2 - r^2)\frac{1}{r+R} + (r+R) \quad \big(u^{-1/2} = 1/(r+R)\big) \\ &= \frac{R^2 - r^2 + r^2 + 2rR + R^2}{r+R} \\ &= \frac{2R(r+R)}{r+R} = 2R. \end{aligned}$$

Similarly :

$$\begin{aligned} f\big((r-R)^2\big) &= (R^2 - r^2)\frac{1}{|r-R|} + |r-R| \\ &= \frac{R^2 - r^2 + r^2 - 2rR + R^2}{r-R} \\ &= \frac{2R(r-R)}{r-R} = 2R. \end{aligned}$$

(Because $r < R$ we can drop the absolute values without changing signs.)

Therefore :

$$f\big((r+R)^2\big) - f\big((r-R)^2\big) = 2R - 2R = 0. \tag{9}$$

No stray minus survives ; the two evaluations simply cancel. And so :

$$\boxed{F_z = 0} \tag{10}$$

Because every other direction is equivalent by symmetry, the total gravitational field at P is zero :

$$\mathbf{g}(P) = \mathbf{0} \qquad \text{for all } r < R.$$

This remarkable result shows that the gravitational field is exactly zero everywhere inside a uniform spherical shell. This is true regardless of where inside the shell we measure the field.

2.2. **Physical interpretation.**

The theorem can be understood by considering that for any point inside a spherical shell, the gravitational attractions from opposite sides of the shell exactly cancel. While closer portions of the shell exert stronger force per unit mass, there is less mass in those closer portions (due to the solid angle effect), and these two effects precisely balance !

3. Revisit integration using the Shell Theorem

We can now get a far simpler derivation for our uniform sphere.

Now the key insight is to use the Shell theorem : we split the sphere into concentric shells and use the fact that :

- A spherical shell with radius $r' > r$ contributes zero field at point P inside it
- A spherical shell with radius $r' < r$ acts as if all its mass were concentrated at the center

Therefore, only the mass within radius r contributes to the field :

$$g(r) = \frac{GM_{\text{enclosed}}}{r^2} = \frac{G \cdot \frac{4\pi r^3 \rho}{3}}{r^2} = \frac{4\pi G\rho r}{3}$$

Since $\rho = \frac{3M}{4\pi R^3}$ for a uniform sphere :

$$\boxed{g(r) = \frac{GMr}{R^3}} \quad \text{for } r \leq R$$

This shows that:

- At $r = 0$: $g = 0$ (as expected by symmetry)
- At $r = R$: $g = \frac{GM}{R^2}$ (matches surface gravity)
- The field varies linearly with r inside the sphere

4. Alternative derivation using Gauss's law

We can also confirm this result using Gauss's law for gravity :

$$\oint_S \vec{g} \cdot d\vec{A} = -4\pi GM_{\text{enclosed}}$$

For a Gaussian surface of radius $r < R$ centered at the sphere's center :

$$g(r) \cdot 4\pi r^2 = -4\pi G \cdot \frac{4\pi r^3 \rho}{3}$$

Solving for $g(r)$ gives the same result.

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[12] Bovy, J., *Dynamics and Astrophysics of Galaxies*, online textbook.
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Email address: `support@visualis-physics.com`

URL: `http://www.visualis-physics.com`