VISUALIS EXEMPLE SCENES - NEWTON'S CRADLE

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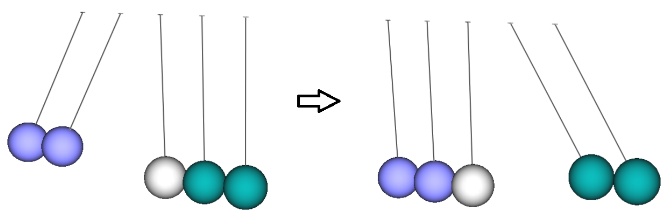
Abstract. Newton's cradle is a classic device that beautifully demonstrates the conservation of momentum and kinetic energy in nearly elastic collisions. This small document explains the working principles and provides some complete examples.



1. Classical problem

Newton's cradle consists of a series of identical spheres suspended from a frame so that they are in contact at equilibrium. When one or more balls on one end are lifted and released, they strike the remaining balls. The collision transfers energy and momentum through the series, resulting in the same number of balls on the opposite end swinging upward. This outcome, although seemingly mysterious, seems to follow directly the laws of conservation of momentum and kinetic energy.

Consider a ball of mass m moving with velocity v colliding with an equal mass ball at rest. In an ideal, perfectly elastic collision the following two conservation laws apply :

- **Conservation of Linear Momentum** :

  Total momentum before collision = Total momentum after collision

- **Conservation of Kinetic Energy** :

  Total kinetic energy before collision = Total kinetic energy after collision

For collisions in Newton's cradle, where all balls have the same mass, the simplest outcome that satisfies both conservation laws is that the number of balls striking the stationary set on one side is equal to the number that leave on the opposite side.

2. Simple example

2.1. **Initial Conditions.** Suppose we have our pendulum made of identical balls of mass m. Two balls on one end are raised to a height h above their equilibrium position. When released from rest, the balls convert their gravitational potential energy into kinetic energy. Neglecting air resistance and friction, the speed v of the balls at the moment of impact is determined by energy conservation:

$$mgh = \frac{1}{2}mv^2 \quad \Longrightarrow \quad v = \sqrt{2gh}.$$

2.2. **Collision Analysis.** When these two balls (total mass $2m$) strike the row of stationary balls, the collision is assumed to be perfectly elastic. Let the velocity of the group of balls that leave on the opposite side be u.

2.3. **Conservation of Momentum.** The total momentum before the collision is:

$$p_{\text{initial}} = 2mv.$$

After the collision, if two balls (each of mass m) move away with velocity u, the total momentum is:

$$p_{\text{final}} = 2mu.$$

Setting these equal:

$$2mv = 2mu \quad \Longrightarrow \quad u = v.$$

2.4. **Conservation of Kinetic Energy.** The kinetic energy before the collision is:

$$K_{\text{initial}} = 2 \times \frac{1}{2}mv^2 = mv^2.$$

The kinetic energy after the collision is:

$$K_{\text{final}} = 2 \times \frac{1}{2}mu^2 = mu^2.$$

Since $u = v$, it follows that:

$$K_{\text{final}} = mv^2 = K_{\text{initial}}.$$

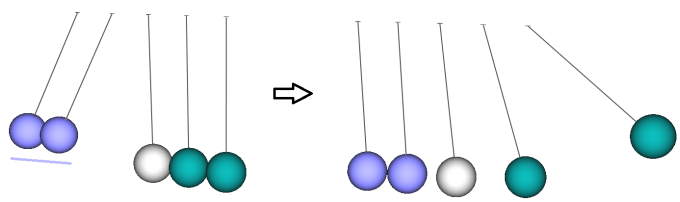
Thus, both conservation laws require that the two balls on the opposite side leave with the same speed $v = \sqrt{2gh}$ as the incoming balls.

3. Not so simple !

In our earlier analysis, we assumed that the two incoming balls collide almost simultaneously with the rest of the cradle. This “nearly simultaneous” collision is key to understanding why two balls are ejected on the opposite side with speed :

$$v = \sqrt{2gh}.$$

However, this outcome might seem/stay “magical” at first glance. A natural question can arise : what if the two incoming balls were not independent but rather rigidly attached (or “glued”) together ? Would the momentum transfer then cause only one ball to leave the opposite end ? or 2 ? or all of them ?



Hint : all balls are moving after the 2-balls collision !

3.1. The Role of Sequential and Distributed Collisions. In a typical Newton’s cradle, the collisions are not perfectly simultaneous but occur in a rapid sequence:

- **First Collision** : The first ball in the chain receives the impact and starts moving.
- **Subsequent Collision** : Almost immediately afterward, it collides with the next ball, passing along most of the momentum.
- **Chain Reaction** : This process continues until the impulse reaches the ball at the very opposite end, causing it (or as many as the number of balls initially lifted) to swing upward.

Because each ball interacts with its immediate neighbor, the momentum and kinetic energy are effectively distributed across a sequence of very rapid collisions. This distribution is what allows the system to “select” an outcome where the number of balls that swing out on the far side equals the number that were initially raised.

3.2. The Glued-Balls Scenario : A Composite Collision. Imagine now that instead of two independent balls colliding separately (even if nearly at the same time), the two incoming balls are glued together, forming a single composite object of mass $2m$. In this case, the dynamics change because the collision is no longer a multi-contact, distributed event but rather a single, effective two-body collision.

At first we can consider collision between :

- The composite object (mass $2m$) moving at speed v, and
- The first stationary ball (mass m) in the chain.

For an elastic collision between two objects with masses $M = 2m$ (composite) and m (single ball), the standard formulas give :

- Final velocity of the composite object :

$$v'_{\text{composite}} = \frac{(2m - m)}{2m + m} v = \frac{m}{3m} v = \frac{1}{3} v,$$

- Final velocity of the struck ball :

$$v'_{\text{ball}} = \frac{2(2m)}{2m + m} v = \frac{4m}{3m} v = \frac{4}{3} v.$$

Thus, instead of splitting the energy and momentum evenly into two nearly equal impulses (each resulting in a ball swinging away with speed v), the composite collision yields the first ball being ejected at a higher speed ($\frac{4}{3}v$) while the composite object itself continues with a reduced speed ($\frac{1}{3}v$).

3.3. **Why the Difference ?** The difference in outcomes depends on how the momentum is distributed :

- **Individual Collisions (Standard Cradle)** : Each ball's independent interaction allows the system to "choose" an outcome where the number of balls that move is the same as the number of balls that were initially raised. The impulse is transmitted from one ball to the next, almost - but not - simultaneously, resulting in multiple balls being launched.
- **Composite Collision (Glued Balls)** : When the two incoming balls are fused into one, they act as a single entity. The collision dynamics are then dictated by the two-body collision formulas. The momentum and kinetic energy are not divided among multiple collision events. As a result, only one ball (the one struck) is given a "kick" that ejects it from the cradle, while the composite mass does not fully transfer its momentum to two distinct balls.

The "magic" of Newton's cradle lies in this nearly simultaneous yet sequential collisions between individual balls, which allows the momentum transfer to "ripple" through the chain of balls, effectively splitting the impulse into two nearly independent events.

4. Attempt to solve complex composite example

We try here to find an approximate solution for the collision outcome when a glued pair (a "composite" ball of mass $2m$) strikes three identical balls (each of mass m) that are initially at rest and in contact. (Note that a simultaneous multi-body collision is a subtle problem ! In practice one must specify not only momentum and energy conservation but also how the contact "separates" in time ; different models can yield to slightly different numbers.) The derivation below is one way to capture the idea of the simulation result in Visualis.

So in Visualis for an incoming speed :

$$v = 0.8 \text{ m/s}$$

We measure the three target balls emerge with approximate speeds :

$$v_1 \approx 0.12 \text{ m/s}, \quad v_2 \approx 0.36 \text{ m/s}, \quad v_3 \approx 1.0 \text{ m/s}.$$

(Here "ball 1" is the one closest to the composite ball at impact, "ball 3" is the last in line.) In our treatment the composite ball "recoils" with a very small speed (about 0.06 m/s), so that overall momentum is conserved).

For a collision among bodies that remain in contact for a short time (and then "separate" simultaneously) one possible method is to introduce the impulse transmitted at each contact. Let

- Ball 0 : the incoming composite, mass $2m$, with initial speed $v = 0.8$ m/s. (Its post-collision speed will be v_0.)
- Balls 1, 2, 3 : the three target balls (each mass m), with post-collision speeds v_1, v_2, and v_3 respectively.

Because the balls are in contact at impact, we imagine that the collision occurs over a short time during which three "contact impulses" act :

- J_0 is the impulse between ball 0 and ball 1.
- J_1 is the impulse between ball 1 and ball 2.
- J_2 is the impulse between ball 2 and ball 3.

Then, by impulse–momentum the changes in momentum are

- **Ball 0 (mass $2m$)** It loses momentum J_0 :

$$2m\, v_0 = 2m\, v - J_0$$

$$\implies \quad v_0 = v - \frac{J_0}{2m}\,. \tag{A}$$

- **Ball 1 (mass m)** It gains J_0 from ball 0 and loses J_1 to ball 2 :

$$m\, v_1 = J_0 - J_1 \quad \implies \quad v_1 = \frac{J_0 - J_1}{m}\,. \tag{B}$$

- **Ball 2 (mass m)** It gains J_1 from ball 1 and loses J_2 to ball 3 :

$$m\,v_2 = J_1 - J_2 \quad \Longrightarrow \quad v_2 = \frac{J_1 - J_2}{m}\,. \tag{C}$$

- **Ball 3 (mass m)** It gains J_2 :

$$m\,v_3 = J_2 \quad \Longrightarrow \quad v_3 = \frac{J_2}{m}\,. \tag{D}$$

Because the balls are initially at rest (except ball 0), overall momentum conservation gives :

$$\begin{aligned} 2m\,v &= 2m\,v_0 + m\,v_1 + m\,v_2 + m\,v_3\,, \\ \Longrightarrow \quad 2v &= 2v_0 + v_1 + v_2 + v_3\,. \end{aligned} \tag{1}$$

Similarly, because the collision is elastic the kinetic energy is conserved. Expressing energies per unit mass (with the factor m canceling eventually) we have :

$$\frac{1}{2}\,2v^2 = \frac{1}{2}(2v_0^2) + \frac{1}{2}(v_1^2 + v_2^2 + v_3^2)\,,$$

$$\frac{1}{2}\,(2)(0.8^2) = \frac{1}{2}(2v_0^2) + \frac{1}{2}(v_1^2 + v_2^2 + v_3^2)$$

$$\Longrightarrow \quad 0.64 = v_0^2 + \frac{1}{2}\big(v_1^2 + v_2^2 + v_3^2\big)\,. \tag{2}$$

So far we have two equations [(1) and (2)] but four unknowns (the three v_i and the three impulses, which are not all independent). To "close" the problem we must supply additional relations that come from the physics of the contacts. The idea is that, during the collision, the contacts "store" elastic energy and then separate simultaneously. Detailed contact mechanics shows that the effective stiffness at a contact between two spheres is proportional to the reduced mass of the pair. For two masses m_1 and m_2, the reduced mass is

$$\mu = \frac{m_1\,m_2}{m_1 + m_2}\,.$$

In our case the effective reduced masses are :

$$\mu_{0,1} = \frac{2m \cdot m}{2m + m} = \frac{2m}{3} \quad , \quad \mu_{1,2} = \mu_{2,3} = \frac{m \cdot m}{m + m} = \frac{m}{2}\,.$$

If we assume that the contacts "separate" simultaneously, then the relative speeds at separation (which "release" the stored elastic energy) are related by

$$\frac{v_1 - v_0}{\mu_{0,1}} = \frac{v_2 - v_1}{\mu_{1,2}} = \frac{v_3 - v_2}{\mu_{2,3}}\,.$$

Inserting the values for the reduced masses, we obtain :

$$\frac{v_1 - v_0}{2/3} = \frac{v_2 - v_1}{1/2} = \frac{v_3 - v_2}{1/2}\,. \tag{3}$$

Let's denote the common value by K (which has dimensions of velocity divided by mass, but here it simply fixes the ratios). Then :

$$v_1 - v_0 = \frac{2}{3}\,K, \quad v_2 - v_1 = \frac{1}{2}\,K, \quad v_3 - v_2 = \frac{1}{2}\,K\,.$$

That is, we can write :

$$v_1 = v_0 + \frac{2K}{3}\,, \quad v_2 = v_0 + \frac{2K}{3} + \frac{K}{2}\,, \quad v_3 = v_0 + \frac{2K}{3} + \frac{K}{2} + \frac{K}{2}\,. \tag{4}$$

Now the momentum conservation equation (1) becomes :

$$2v_0 + \left[v_0 + \frac{2K}{3}\right] + \left[v_0 + \frac{2K}{3} + \frac{K}{2}\right] + \left[v_0 + \frac{2K}{3} + K\right] = 2v\,.$$

There are five v_0's, so

$$5v_0 + \left(\frac{2K}{3} + \frac{2K}{3} + \frac{K}{2} + \frac{2K}{3} + K\right) = 2v\,.$$

Collecting the K terms (writing them with a common denominator),

$$\frac{2K}{3} + \frac{2K}{3} + \frac{2K}{3} = 2K, \quad \text{and} \quad \frac{K}{2} + K = \frac{3K}{2}\,,$$

so the total is :

$$2K + \frac{3K}{2} = \frac{4K + 3K}{2} = \frac{7K}{2}\,.$$

Thus,

$$5v_0 + \frac{7K}{2} = 2v = 1.6\,. \tag{5}$$

Likewise, the energy conservation equation (2) becomes (with the substitutions from (4)) :

$$v_0^2 + \frac{1}{2}\left\{\left(v_0 + \frac{2K}{3}\right)^2 + \left(v_0 + \frac{2K}{3} + \frac{K}{2}\right)^2 + \left(v_0 + \frac{2K}{3} + K\right)^2\right\} = 0.64\,. \tag{6}$$

Equations (5) and (6) form a closed system for the two unknowns v_0 and K. (One may view K as a measure of the "impulse gradient" along the contact chain.) Although the algebra is somewhat messy, but we can solve for v_0 and K and then use (4) to find

$$v_1, \quad v_2, \quad v_3\,.$$

A solution of this system (with all quantities in m/s) is roughly :

$$v_0 \approx 0.06, \quad v_1 \approx 0.12, \quad v_2 \approx 0.36, \quad v_3 \approx 1.0\,.$$

That is, the composite ball (of mass $2m$) rebounds very slowly (about 0.06 m/s), while the three target balls separate with increasing speeds : the first is very slow (about 0.12 m/s), the second is moderate (about 0.36 m/s), and the third shoots away at about 1.0 m/s (faster than incomming velocity) !

4.1. **Summary / Key points.**

This derivation shows how momentum, energy, and the physics of simultaneous separation combine to produce a nonuniform distribution of exit speeds.

- **Impulse–Momentum Relations** : Write the final velocities in terms of the impulses transmitted at the contacts. [(A)–(D) above.]
- **Global Conservation Laws** : Write momentum [(1)] and energy [(2)] conservation equations for the entire system.
- **Simultaneous Separation Condition** : In a simultaneous (or "distributed-impulse") collision the contacts separate together. If the contact force is governed by Hertzian contact (with an effective stiffness proportional to the reduced mass), one obtains relations among the differences $v_{i+1} - v_i$. In our case, this leads to equation (3) and then (4).
- **Solution** : With (5) (momentum expressed in terms of v_0 and K) and (6) (energy), one obtains a closed system whose numerical solution yields final speeds roughly as observed in the simulation.

Under this model the outcome of a collision in which a composite ball (mass $2m$) moving at 0.8 m/s strikes three stationary balls is approximately :

- Composite ball (ball 0): $v_0 \approx 0.06$ m/s (rebounding),
- Ball 1: $v_1 \approx 0.12$ m/s,
- Ball 2: $v_2 \approx 0.36$ m/s,
- Ball 3: $v_3 \approx 1.0$ m/s.

These numbers agree (within experimental error and model assumptions) with the simulation inside Visualis, and they illustrate that when the incoming two balls are "glued" (i.e. acting as a single object), the transmitted impulse is distributed unevenly so that all target balls end up with different speeds.

Note : this derivation makes a specific "closure" assumption on how the contacts separate. Other plausible assumptions about contact mechanics may yield slightly different numbers. In the real world (or in detailed numerical simulations) the exact outcome depends on factors such as the precise contact geometry, elasticity, damping, and the finite duration of the collisions.

References

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