VISUALIS EXAMPLE SCENES - VIBRATING STRING

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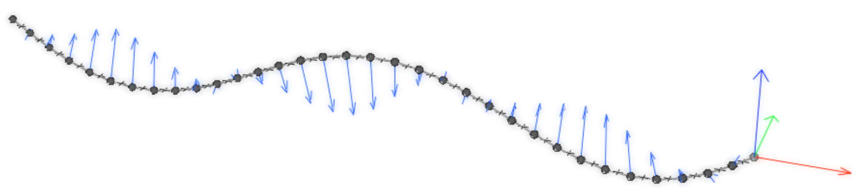
Abstract. This document provide a summary of basic maths for the study of a vibrating string (Melde's Experiment) in Visualis Dynamics.

1. Vibrating String - Melde's Experiment

This experiment, commonly known as Melde's Experiment, shows how a stretched cord can produce stable, stationary patterns when forced to vibrate at specific frequencies.



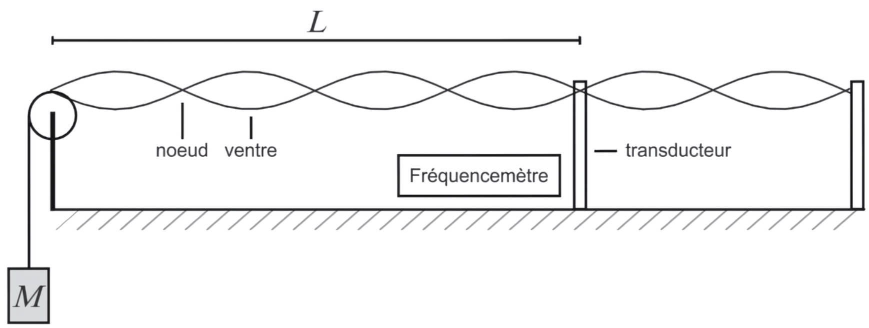
Oscillating rope in Visualis, showing force vectors

The formation of a standing wave on a tensioned string illustrates the principle of mechanical resonance through the constructive interference of opposing wave trains. By subjecting one end of the cord to a periodic driving force while the other remains fixed or under constant tension, reflected waves superimpose upon incident ones ; when the driving frequency aligns with a natural harmonic of the medium, the system achieves a steady state characterized by stationary nodes and maximal-amplitude antinodes ! This phenomenon effectively traps energy within the string's spatial constraints, transforming a traveling disturbance into a stable, oscillating geometry where the phase relationship remains fixed in time.

2. Physical setup

We consider a rope :

- of length L,
- under a uniform tension T,
- with linear mass density μ (mass per unit length),
- fixed at both ends : $x = 0$ and $x = L$.



Schema of experiment at Nyon's high school

We consider transverse oscillations $y(x,t)$ ($\Rightarrow$ vertical displacement and horizontal rope).

(note : gravity is neglected here, as it operates on a different scale .. it slightly changes the curvature of a long rope, but does not affect the principle or the phenomenon of resonance.)

For small angles, the rope obeys the **1-D wave equation** :

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2} \quad \text{where} \quad v = \sqrt{\frac{T}{\mu}}$$

is the speed of transverse waves on the rope.

The fixed ends impose the boundary conditions :

$$y(0,t) = 0, \qquad y(L,t) = 0 \quad \text{for all } t.$$

Note : In the real experiment (and in Visualis) one end is not strictly fixed : the shaker moves it with a small amplitude a_{drive}, so the exact boundary condition there is $y(0,t) = a_{\text{drive}} \cos(\omega t)$ rather than $y(0,t) = 0$. However, at resonance the antinode amplitude grows much larger than a_{drive}. On the scale of the standing wave, the driven end therefore behaves as a *quasi-node* : it moves just enough to inject energy into the rope, while remaining almost motionless compared to the antinodes. The idealized condition $y(0,t) = 0$ is thus an excellent approximation for finding the mode shapes.

3. Travelling waves and standing waves

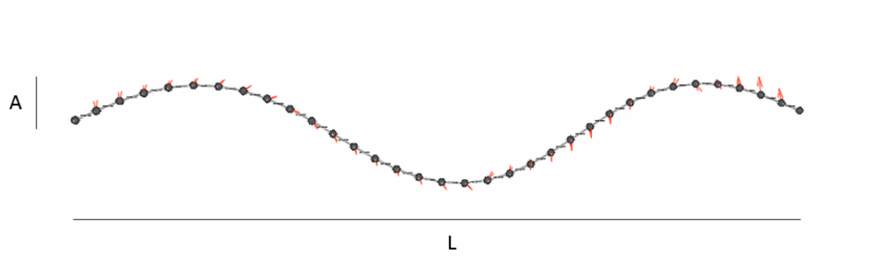
If we shake one end periodically, we create travelling waves that propagate along the rope :

- A wave travelling to the right : $y_{\text{right}}(x,t) = A\sin(kx - \omega t)$,
- A wave travelling to the left : $y_{\text{left}}(x,t) = A\sin(kx + \omega t)$,

with

$$\omega = 2\pi f, \qquad k = \frac{2\pi}{\lambda}, \qquad v = \frac{\omega}{k} = f\lambda.$$

If both left and right-going waves of same amplitude and frequency exist, their superposition gives a **standing wave** !



$$\begin{aligned} y(x,t) &= y_{\text{right}}(x,t) + y_{\text{left}}(x,t) \\ &= A\sin(kx - \omega t) + A\sin(kx + \omega t) \\ &= 2A\sin(kx)\cos(\omega t). \end{aligned} \tag{1}$$

The key point : in a standing wave,

- the spatial shape is given by $\sin(kx)$,
- the time dependence is just an overall oscillation $\cos(\omega t)$.

Certain points, where $\sin(kx) = 0$, almost don't move ! these are the **nodes**, and points where $|\sin(kx)|$ is maximal move the most are the **antinodes**.

4. Allowed modes of a rope fixed at both ends

Now we impose the fixed-end conditions on a general standing wave :

$$y(x,t) = \Phi(x)\cos(\omega t + \phi),$$

with $\Phi(x)$ a spatial function.

We can find all possible shapes $\Phi(x)$ by solving the wave equation using separation of variables.

4.1. **Separation of variables.**

We can assume a solution of the form :

$$y(x,t) = X(x)T(t).$$

At resonance every point of the rope oscillates at the same frequency, and only the amplitude depends on the position. So the spatial shape is "frozen" : it only scales up and down in time. This is precisely what a product $X(x)T(t)$ expresses, a fixed shape $X(x)$ multiplied by a global oscillation $T(t)$. And from a mathematical point of view : the wave equation is linear, so any superposition of solutions is again a solution. The separated solutions form a complete family (a Fourier basis) : any motion of the rope can be written as a sum of them. Looking for product solutions therefore loses no generality.

So plug $y(x,t) = X(x)T(t)$ into the wave equation :

$$X(x)T''(t) = v^2 X''(x)T(t).$$

Divide both sides by $v^2 X(x)T(t)$ (for nonzero X, T) :

$$\frac{T''(t)}{v^2 T(t)} = \frac{X''(x)}{X(x)} = -k^2,$$

where $-k^2$ is a separation constant (negative so that we get oscillatory solutions). This gives two ODEs :

- Time part :

$$T''(t) + \omega^2 T(t) = 0, \quad \text{with } \omega = vk,$$

- Space part :

$$X''(x) + k^2 X(x) = 0.$$

The general spatial solution is

$$X(x) = A\sin(kx) + B\cos(kx).$$

4.2. **Boundary conditions, quantisation of (k).**

Apply $y(0,t) = 0$. That is $X(0)T(t) = 0$ for all t, so $X(0) = 0$:

$$X(0) = A\sin(0) + B\cos(0) = B = 0.$$

So $B = 0$ and

$$X(x) = A\sin(kx).$$

Apply $y(L,t) = 0 \Rightarrow X(L) = 0$:

$$X(L) = A\sin(kL) = 0.$$

For a nontrivial amplitude $A \neq 0$, we must have

$$\sin(kL) = 0 \quad \Rightarrow \quad kL = n\pi, \quad n = 1, 2, 3, \ldots$$

So the rope only supports discrete values

$$k_n = \frac{n\pi}{L}, \qquad n = 1, 2, 3, \ldots$$

and thus discrete angular frequencies

$$\omega_n = vk_n = v\frac{n\pi}{L}.$$

The corresponding frequencies are

$$f_n = \frac{\omega_n}{2\pi} = \frac{vk_n}{2\pi} = \frac{nv}{2L} = \frac{n}{2L}\sqrt{\frac{T}{\mu}}, \qquad n = 1, 2, 3, \ldots$$

These are the **resonant frequencies** or **normal mode frequencies**. They form a harmonic series: $f_n = nf_1$ where

$$f_1 = \frac{v}{2L} = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

is the fundamental frequency.

5. Shape of the modes, nodes and antinodes

The n-th normal mode has the form :

$$y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t + \phi_n),$$

with

$$\omega_n = \frac{n\pi v}{L}, \qquad f_n = \frac{\omega_n}{2\pi}.$$

5.1. **Nodes.** Nodes occur where the spatial factor is zero :

$$\sin\left(\frac{n\pi x}{L}\right) = 0 \quad \Rightarrow \quad \frac{n\pi x}{L} = m\pi \quad \Rightarrow \quad x_m = \frac{mL}{n},$$

with $m = 0, 1, 2, \ldots, n$.

So :

- There are (n+1) nodes in total, including the two fixed ends.
- There are (n-1) internal nodes (strictly between 0 and L).

This matches what we see :

- For the fundamental (n=1): only the two endpoints are nodes (no internal node).
- For (n=2) : one internal node at $x = L/2$.
- For (n=3) : two internal nodes at $x = L/3$ and $2L/3$.
- etc ..

5.2. **Antinodes.** Antinodes occur where $|\sin(n\pi x/L)|$ is maximal $= 1$, i.e. halfway between nodes.

For the n-th mode :

$$x_{\text{antinodes}} = \frac{(2m-1)L}{2n}, \quad m = 1, 2, \ldots, n.$$

So the n-th mode has exactly n antinodes.

6. Relation to wavelength and "1, 2, 3 nodes" result

The wavelength λ is related to k by $k = 2\pi/\lambda$. For mode (n) :

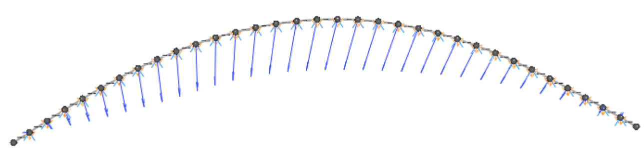
$$k_n = \frac{2\pi}{\lambda_n} = \frac{n\pi}{L} \quad \Rightarrow \quad \lambda_n = \frac{2L}{n}.$$

So the rope of length L contains :

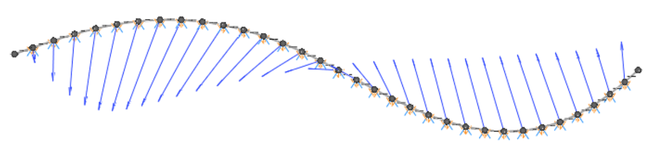
- half a wavelength for the fundamental $n = 1$: $\lambda_1 = 2L$,
- one full wavelength for the 2nd harmonic $n = 2$: $\lambda_2 = L$,
- one and a half wavelengths for the 3rd, etc.

Visually :

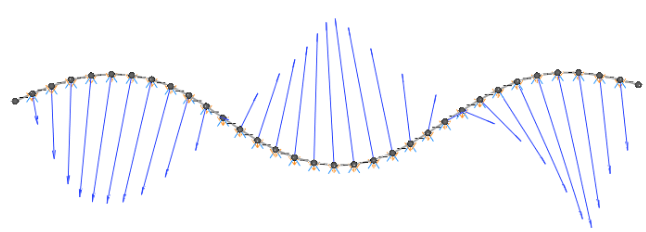
- **Fundamental** ($n = 1$) Shape : one "big bump" ; nodes at (x=0) and (x=L). Internal nodes : 0.



- **2nd harmonic** ($n = 2$) Shape : two "bumps" ; nodes at (x=0, L/2, L). Internal nodes : 1.



- **3rd harmonic** ($n = 3$) Shape : three "bumps" ; nodes at (x=0, L/3, 2L/3, L). Internal nodes : 2.



7. Driving the rope and resonance

In the experiment, we vibrate one end (or use an attached vibrator) at some driving frequency f_{drive}. The response of the rope depends strongly on the relation between f_{drive} and the natural frequencies f_n.

7.1. **Off-resonance.**

If f_{drive} is not close to any f_n, the rope still oscillates, but :

- the pattern is not clean,
- energy reflects back and forth but doesn't accumulate coherently,
- the amplitude stays moderate and the shape is messy / time-dependent.

We see waves travelling along the rope, but no stable node pattern.

7.2. **Resonance.**

When f_{drive} is very close to one of the natural frequencies f_n, we get **resonance** :

$$f_{\text{drive}} \approx f_n = \frac{n}{2L}\sqrt{\frac{T}{\mu}}.$$

Then :

- The waves travelling back and forth interfere **constructively**.
- The rope settles into the standing wave pattern of that mode (n).
- Nodes become very clearly visible : points that barely move.
- Antinodes oscillate with large amplitude.

In Visualis by adjusting either :

- the frequency f of the shaker, or
- the tension T (thus changing v and all f_n),

we can easily “scan” through the modes :

- First we see the fundamental pattern (no internal nodes).
- Then, at a higher frequency, a pattern with 1 internal node.
- Then with 2 internal nodes, etc.

8. Visualis Simulation

In Visualis, the rope is not simulated by directly solving the analytical wave equation. Instead, it is represented as a simple mechanical system : a chain of mass points connected by elastic segments.

This means that the rope is discretized into many small particles :

$$P_0, P_1, P_2, \ldots, P_N,$$

where neighbouring particles are connected by constraints or spring-like segments. Each point has a position, a velocity, a mass, and is affected by forces. The two end points can be fixed, or one end can be periodically moved to act as a shaker.

For each simulation step, Visualis applies Newton's law to every point :

$$\vec{F}_i = m_i \vec{a}_i \quad \text{(or equivalently } \vec{a}_i = \tfrac{\vec{F}_i}{m_i}\text{)}.$$

The forces mainly come from the neighbouring segments. If a segment is stretched, it pulls the two connected points back together ; if it is compressed, it pushes them apart. In a simplified spring model, the force between two points can be written as

$$\vec{F}_{i,i+1} = -k \left(|\vec{P}_{i+1} - \vec{P}_i| - l_0 \right) \frac{\vec{P}_{i+1} - \vec{P}_i}{|\vec{P}_{i+1} - \vec{P}_i|},$$

where :

- k is the stiffness of the segment,
- l_0 is the rest length of the segment,
- $|\vec{P}_{i+1} - \vec{P}_i|$ is the current length of the segment.

A small damping term may also be added, so that the system does not accumulate unlimited energy :

$$\vec{F}_{\text{damping}} = -c\vec{v}_i.$$

The motion is then integrated numerically. A common method for this kind of simulation is Verlet integration, which updates positions directly from their current and previous positions :

$$\vec{P}_i(t + \Delta t) = 2\vec{P}_i(t) - \vec{P}_i(t - \Delta t) + \vec{a}_i(t)\Delta t^2.$$

This is very simple, but surprisingly powerful. The rope is only a collection of points and segments, yet the global behaviour looks like a continuous string.

8.1. **Emergence of wave behaviour.**

The important point is that Visualis does not explicitly "create" a sine wave, a wavelength, a node, or a harmonic mode.

These structures emerge from the local interactions between neighbouring points.

When one point is displaced, it pulls on the next point, which pulls on the next one, and so on. The disturbance therefore propagates along the rope. In the continuous limit, when the number of points becomes large and the segments are short, this discrete chain behaves like a real string and approaches the 1-D wave equation :

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}.$$

In other words, the wave equation is not imposed from above. It appears as the macroscopic limit of many small Newtonian interactions.

This is one of the most interesting aspects of physics simulation : complex phenomena will arise from very simple rules. A rope does not need to "know" anything about standing waves. Each point only follows Newton's laws and reacts to its neighbours. Nevertheless, when the whole chain is simulated, waves propagate, reflect, interfere, and sometimes organize themselves into stable standing wave patterns.

8.2. **Why standing waves appear.**

When the rope is fixed at both ends, any travelling wave is reflected when it reaches an end. The reflected wave travels back in the opposite direction and overlaps with the incoming wave.

If the driving frequency is arbitrary, the reflected waves usually do not return with the correct phase. The shape remains irregular and time-dependent.

But when the driving frequency is close to one of the natural frequencies of the chain, the reflected waves come back in phase with the new waves produced by the shaker. The same pattern is reinforced again and again. Energy is injected coherently into the rope, and the motion progressively organizes into a normal mode.

This produces the same condition as in the analytical theory :

$$L = n\frac{\lambda_n}{2},$$

or

$$\lambda_n = \frac{2L}{n}.$$

The nodes appear because at some positions the waves travelling in opposite directions cancel each other at all times. These points receive contradictory motions from both sides and remain almost motionless.

Similarly, the antinodes appear where the two waves reinforce each other, creating large oscillations.

So the familiar patterns are emergent structures generated by the dynamics of the simulated rope.

8.3. **Discrete simulation and real physics.**

Because the Visualis rope is made of a finite number of points, it is not a perfect mathematical continuum. The number of points, the stiffness of the segments, the time step, damping, and constraint solver all influence the result.

However, this is not really a weakness. It is also what happens in real materials : a real rope is not truly continuous either. It is made of fibres, molecules, and microscopic structures. The continuous wave equation is itself an approximation of a more detailed physical reality.

The simulation therefore gives an intuitive bridge between two descriptions :

- the microscopic or numerical description : points, masses, forces, constraints, time steps ;
- the macroscopic description : waves, wavelength, resonance, harmonics, nodes and antinodes.

Simulators like Visualis makes this bridge clearly visible. By increasing the driving frequency, changing the tension, or modifying the rope parameters, we can directly observe how a simple Newtonian system leads to the classical standing wave modes of Melde's experiment.

This is why the result is always fascinating : the simulation contains no special rule for standing waves. They simply appear.

References

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